

## GFD I, Final Exam, 3/7/2012

*Due in my mailbox in OSB by Noon, Wednesday 3/14/2012.*

You may consult your class notes, Vallis, Holton, Gill, and Kundu & Cohen, but you should not discuss this with your classmates. Each problem is worth 30 points.

1. Consider linear, inviscid, Boussinesq,  $f$ -plane flow of a linearly-stratified flow over corrugated hills with shape  $z_b = \delta \sin[k(x - Ut)]$ . This is the IGW “flow-over-topography” problem in the frame of reference in which the hills are moving under an atmosphere which is stationary (except for the wave perturbations), exactly as done in class. Assume that the frequency is high enough so that the waves are in the “non-hydrostatic, non-rotating” regime, exactly as in problem 2 of Problem Set 5.

- (a) Derive the expression for the pressure perturbation,  $p'/\rho_0$ .
- (b) Rewrite your result from (a) giving the amplitude of the pressure perturbations in terms of the parameters  $N$ ,  $\delta$ ,  $U$ , and the ratio “ $F$ ” of the intrinsic frequency,  $Uk$ , to the buoyancy frequency:  $F \equiv Uk/N$ .
- (c) What happens to the perturbation pressure as non-hydrostatic effects become important?
- (d) Can you come up with a simple physical explanation for the result in (c)?
- (e) Will the sign and phasing of your pressure be likely to give rise to form drag? Note that the expression for the form drag is given by:

$$\frac{\text{Force exerted on the fluid by the boundary}}{\text{unit horizontal area}} = -\overline{p' \Big|_{z=0} \frac{\partial z_b}{\partial x}}^x$$

where the overbar denotes  $x$ -averaging. Check that the sign of your result makes sense physically.

2. This is an extension of the Ekman spin-down problem we did in class. I would like you to consider how the energy budget works for this problem. The Boussinesq equation we for energy conservation may be written as:

$$\frac{D}{Dt}(KE_v + PE_v) = -\nabla \cdot (\mathbf{u}p)$$

In class we worked out what these terms looked like for linear SWE flow, after integrating vertically over the fluid thickness. Consider a volume integral of this equation, where the volume of integration,  $V$ , extends vertically from just above the top of the Ekman layer to the free surface of the fluid. The bottom of the fluid is at  $z = 0$ , the free surface of the fluid is at  $z = H + \eta$ , and we will define the position just above the top of the bottom boundary layer as being  $z = \delta^+$ . You may assume that the Ekman layer thickness is much smaller than the resting fluid thickness,  $H$ . The volume integral of the left hand side is given approximately by

$$\int_V \frac{D}{Dt} (KE_V + PE_V) dV \cong \frac{\partial}{\partial t} \int_V (KE_V + PE_V) dV$$

On the right-hand side the “pressure work” may be rewritten, using Gauss’ Divergence Theorem, as

$$\int_V [-\nabla \cdot (\mathbf{u}p)] dV = - \int_A u_n p dA$$

That is, the volume-integrated pressure work convergence equals the area integral of the pressure times the velocity normal to the bounding area of the integral (defined as  $u_n$ , where a normal velocity *out* of the volume is positive). The pressure in the integral is evaluated on the bounding surface. The only part of the bounding surface of our volume where the pressure work term will be important is on the lower area, the horizontal plane just above the bottom boundary layer. Thus we would like to understand both the pressure at the bottom of this volume, and the vertical velocity through it.

(a) Integrate the mass-conservation equation,  $\nabla \cdot \mathbf{u} = 0$ , vertically from the actual physical bottom boundary of the fluid, up to a position just above the Ekman layer thickness,  $z = \delta^+$ . Note that  $\partial/\partial x = 0$  from the physical set-up of this problem. From your integral of the MASS equation, please show how the vertical velocity at  $z = \delta^+$  is related functionally to the horizontal divergence of the Ekman transport in the bottom boundary layer. The vertical velocity at  $z = \delta^+$  is called the “Ekman pumping velocity”  $w_E$ . Note that in class we integrated from  $z = 0$  to  $z = \infty$  to define the Ekman transport, but you can assume that it is about the same as the transport when the upper limit of integration is just above the boundary layer, as we are supposing here.

- (b) How is  $w_E$  functionally related to the overlying velocity field,  $U(y,t)$ ?
- (c) What is the pressure just above the top of the boundary layer? You may assume it is hydrostatic.
- (d) What is the expression for the pressure work term from the volume integral of the overlying fluid energy equation,

$$-\int_A (u_n p) dA = -\int_A (u_n p)|_{z=\delta^+} dA = \int_0^L \left[ \int_0^{2\pi/l} (w_E p)|_{z=\delta^+} dy \right] dx?$$

We are assuming that the horizontal limits of integration are over some arbitrary distance  $L$  in the  $x$ -direction, and over one wavelength in the  $y$ -direction.

- (e) Does the pressure work term *add* or *remove* mechanical energy from the volume of fluid above the bottom boundary layer? Does your result make sense physically?

3. Consider a boundary-trapped atmospheric Kelvin wave of amplitude 400 m propagating north up the west coast of North America and leaning against coastal mountains. It is trapped on an 800 m deep “marine inversion” with a 10 K potential temperature jump. This can be treated mathematically as a two-layer problem with an infinitely-thick upper layer. Then the dynamics in the lower layer are governed by our familiar 1-layer Shallow Water Equations but with the reduced gravity in place of gravity. Velocities in the upper layer are negligibly small because the layer is so thick.

(a) Calculate the reduced gravity. For the Density difference you should use the *potential* density jump, and note that for dry air  $\rho_{pot} = p_{ref} / (R\theta)$ , and you may use 1000 mb ( $10^5$  Pa) for the reference pressure and 275 K for the potential temperature in the lower layer. Note that  $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$ .

- (b) Calculate the effective depth,  $H_{eff}$ .
- (c) Calculate the Kelvin wave phase speed.
- (d) Calculate the internal Rossby radius of deformation (assume  $f$  is calculated at 45 degrees North latitude)
- (e) Calculate the maximum northward wind speed associated with the wave.